$$R = \text{span}_{\mathbb{R}}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\} \cong \mathbb{R}^n$$

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} \qquad \varepsilon_i \longmapsto \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$$

$$\mathfrak{h}^*_{\mathbb{R}} = \text{span}_{\mathbb{R}}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

$$R = \mathbb{R}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) \oplus \underbrace{\left(\mathbb{R}(\varepsilon_1 + \cdots + \varepsilon_n)\right)^{\perp}}_{\overset{?}{=}\ \mathfrak{h}^*_{\mathbb{R}}}$$

$$\mathfrak{g} = \mathfrak{sl}(n, F)$$

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le n\}$$

$$\alpha = \varepsilon_i - \varepsilon_j$$

$$\|\alpha\|^2 = (\alpha, \alpha) = (\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j) = 2$$

$$\Rightarrow \|\alpha\| = \sqrt{2}.$$

$\alpha, \beta \in \Delta$, $\theta$ is the angle between $\alpha$ & $\beta$.

$$(\alpha, \beta) = \|\alpha\| \|\beta\| \cos\theta, \quad 0 \le \theta \le \pi.$$

<u>$n=3$</u>: $\mathfrak{g} = \mathfrak{sl}(3, F)$

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3$$

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le 3\}$$

$$= \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}$$

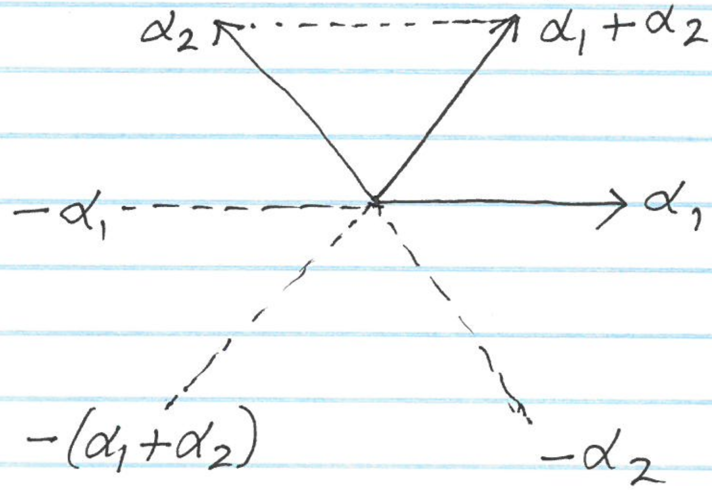
$$= \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$$

$$(\alpha_1, \alpha_2) = (\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3) = -1$$

$$\| \qquad$$

$$\|\alpha_1\|\|\alpha_2\| \cos\theta = (\sqrt{2})(\sqrt{2})\cos\theta.$$

$$\cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$




$$(\alpha_1, \alpha_1+\alpha_2) = (\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3) = 1$$

$$\| \qquad$$

$$\|\alpha_1\|\|\alpha_1+\alpha_2\| \cos\theta = 2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3.$$

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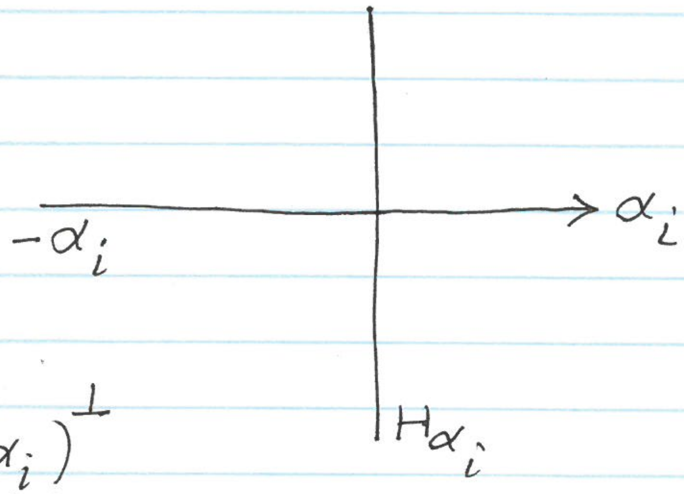
Define $r_i : \mathfrak{h}^*_{\mathbb{R}} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$

by $r_i(\beta) = \beta - \beta(h_i)\alpha_i$

$1 \le i \le n-1.$

$\alpha_i \in \Delta \quad , \quad 1 \le i \le n-1$

$$r_i(\alpha_i) = \alpha_i - \alpha_i(h_i)\alpha_i = \alpha_i - 2\alpha_i = -\alpha_i$$




$$R = (\mathbb{R}\alpha_i) \oplus (\mathbb{R}\alpha_i)^{\perp}$$

$$\lambda \in (\mathbb{R}\alpha_i)^{\perp} = H_{\alpha_i} \Rightarrow \underbrace{(\lambda, \alpha_i)}_{= \lambda(h_i)} = 0$$

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$$
$$= \lambda \qquad \Rightarrow r_i^2 = id$$

$\{r_1, r_2, \cdots, r_{n-1}\}$ simple reflections.

$W$ = the group generated by
$\{r_1, r_2, \cdots, r_{n-1}\}$
called the Weyl group.

$$r_i(\alpha_i) = r_i(\varepsilon_i - \varepsilon_{i+1}) = \varepsilon_{i+1} - \varepsilon_i = -\alpha_i$$

$$r_k(\varepsilon_i - \varepsilon_j) = (\varepsilon_i - \varepsilon_j) - (\varepsilon_i - \varepsilon_j)(h_k)\alpha_k .$$

$$= \cancel{(\varepsilon_i - \varepsilon_j)}$$

$$(\varepsilon_i-\varepsilon_j)(h_k)=(\varepsilon_i-\varepsilon_j)(E_{k,k}-E_{k+1,k+1})$$

$$=\begin{cases} 2 & , \ i=k,\ j=k+1 \\ 1 & \begin{cases} i\neq k,\ j=k+1 \\ i=k,\ j\neq k+1 \end{cases} \\ -2 & i=k+1,\ j=k \\ -1 & \begin{cases} i\neq k+1,\ j=k \\ i=k+1,\ j\neq k \end{cases} \\ 0 & i\neq k,k+1,\ j\neq k,k+1 \end{cases}$$

$$r_k(\varepsilon_i-\varepsilon_j)=\begin{cases} \varepsilon_i-\varepsilon_j & , \ i\neq k,k+1,\ j\neq k,k+1 \\ \varepsilon_j-\varepsilon_i & , \ i=k,\ j=k+1 \\ \varepsilon_i-\varepsilon_k & , \ i\neq k,\ j=k+1 \\ \varepsilon_{k+1}-\varepsilon_j & , \ i=k,\ j\neq k+1 \\ \varepsilon_k-\varepsilon_{k+1} & , \ i=k+1,\ j=k \\ \varepsilon_i-\varepsilon_{k+1} & , \ i\neq k+1,\ j=k \\ \varepsilon_k-\varepsilon_j & , \ i=k+1,\ j\neq k \end{cases}$$

$\Rightarrow \quad r_k \equiv (k, k+1)$ transposition.

$\Rightarrow W = \{(1,2),(2,3),\cdots,(n-1,n)\}$

$=$ generated by $\{(1,2),(2,3),\cdots,(n-1,n)\}$

$= S_n$.

# Representation of $\mathfrak{g} = sl(n, F)$.

$\rho : \mathfrak{g} \longrightarrow End(V)$ finite dim'l representation.

$\Rightarrow$ $V$ is a finite dim'l $\mathfrak{g}$-module

$\Rightarrow$ By Weyl's Thm of complete reducibility that

$V = V_1 \oplus \cdots \oplus V_k$, $V_j$ irred. submod.

$\mathfrak{h} = span\{E_{i,i} - E_{i+1,i+1} = h_i \mid 1 \leq i \leq n-1\}$

Cartan subalg. of $\mathfrak{g}$.

$\rho|_{\mathfrak{h}} : \mathfrak{h} \longrightarrow End(V) = gl(V)$

$V$ is an $\mathfrak{h}$-module.

For $\lambda \in \mathfrak{h}^*$,

$$V_\lambda = \{v \in V \mid h.v = \lambda(h)v \ \forall h \in \mathfrak{h}\}$$

$\lambda$ is called a <u>weight</u> if $V_\lambda \neq 0$, and $V_\lambda$ is called the $\lambda$-weight space.

<u>Defn</u>: (1) A nonzero vector $v \in V$ is a weight vector with weight $\lambda \in \mathfrak{h}^*$ if $h \cdot v = \lambda(h) v \quad \forall\, h \in \mathfrak{h}$.

(2) Let $\lambda \in \mathfrak{h}^*$ be a weight and $0 \neq v \in V_\lambda$. Then $v$ is a weight vector with weight $\lambda$. The dimension of the $\lambda$-weight space $V_\lambda$ is called the <u>multiplicity of $\lambda$</u>.

<u>Example</u>: $ad : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$ repn.
$\Rightarrow \mathfrak{g}$ is a $\mathfrak{g}$-module under adjoint action. Then $\mathfrak{g}$ is a $\mathfrak{h}$-module and for $\lambda \in \mathfrak{h}^*$,

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid h.x = \lambda(h) x \ \forall\, h \in \mathfrak{h}\}$$
$$= \{x \in \mathfrak{g} \mid [h, x] = \lambda(h) x \ \forall\, h \in \mathfrak{h}\}$$
$$\neq 0$$

$\Longrightarrow \lambda = 0$ or $\lambda$ is a root of $\mathfrak{h}$ on $\mathfrak{g}$.

$$\Delta = \Delta_+ \cup \Delta_-$$

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \;, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_0 = \{ x \in \mathfrak{g} \mid [h, x] = 0 \ \forall\, h \in \mathfrak{h} \}$$

$= \mathfrak{h}$ since $\mathfrak{h}$ is a maximal abelian subalgebra.

$$\Longrightarrow \mathfrak{g} = \Big( \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \Big) \oplus \mathfrak{g}_0 \oplus \Big( \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \Big)$$

(root space decomp)

$= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (triangular decomp.)