$\mathfrak{g} = \mathfrak{sl}(n, F)$, $\operatorname{char} F = 0$

A_{n-1} $\qquad \mathbb{R} \subseteq F$

$$A = (a_{ij})_{(n-1)\times(n-1)}, \quad a_{ij} = \alpha_j(h_i) = \begin{cases} 2, & i=j \\ -1, & i=j\pm 1 \\ 0, & \text{otherwise} \end{cases}$$

called the Cartan matrix.

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{2\times 2}, \quad A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}_{3\times 3}$$

$\det(A_2) = 3$, $\det(A_3) = 4$

In general, $\det(A_{n-1}) = n$.

Note: A_{n-1} is positive definite.

$$\mathfrak{h}^* = \operatorname{span}_F\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$$
$$= \operatorname{span}_F\{\omega_1, \omega_2, \cdots, \omega_{n-1}\}$$

$\alpha_j = \varepsilon_j - \varepsilon_{j+1}$, $\omega_j = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_j$

$\alpha_j(h_i) = a_{ij}$, $\omega_j(h_i) = \delta_{ij}$

$n=3$: $\mathfrak{sl}(3, F) = \mathfrak{g}$

$\{\alpha_1, \alpha_2\}$, $\{\omega_1, \omega_2\}$, $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$$A_2^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

$\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = -\omega_1 + 2\omega_2$

$\omega_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$, $\omega_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$

$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ root lattice

$P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ weight lattice

$Q \subset P$, $P \subsetneq Q$

$P/Q = \{Q, Q+\omega_1, Q+\omega_2\}$

$\omega_2 + \omega_1 = \alpha_1 + \alpha_2 \in Q \Rightarrow Q+\omega_2 = Q+(-\omega_1)$
$\neq Q+\omega_1$

$\Rightarrow Q+\omega_2 \neq Q+\omega_1$

Observe

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

n arbitrary case : $sl(n, F)$

$$\det(A_{n-1}) = n$$

$$\mathfrak{h}^* = \operatorname{span}_F\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$$

$$= \operatorname{span}_F\{\omega_1, \omega_2, \cdots, \omega_{n-1}\}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = A_{n-1} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n-1} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n-1} \end{pmatrix} = A_{n-1}^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

$$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \cdots \oplus \mathbb{Z}\alpha_{n-1}$$

root lattice

$$P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \cdots \oplus \mathbb{Z}\omega_{n-1}$$

weight lattice

$$Q \subset P$$

$$P/Q = \{Q, Q+\omega_1, Q+\omega_2, \cdots, Q+\omega_{n-1}\}$$

$$|P/Q| = n.$$

Note A_{n-1} is symmetric since

$$\alpha_j(h_i) = \alpha_i(h_j).$$

Chevalley Relations:

$$[h_i, h_j] = 0 \quad \forall\, i,j$$

$$[h_i, e_j] = \alpha_j(h_i)\, e_j$$

$$[h_i, f_j] = -\alpha_j(h_i)\, f_j$$

$$[e_i, f_j] = \delta_{ij}\, h_i$$

$$\operatorname{ad} e_i(e_j) = [e_i, e_j]$$

$$= [E_{i,i+1}, E_{j,j+1}] = \begin{cases} E_{i,i+2}, & i=j-1 \\ -E_{i-1,i+1}, & i=j+1 \\ 0, & \text{otherwise} \end{cases}$$

$$(\operatorname{ad} e_i)^2(e_j) = \begin{cases} [E_{i,i+1}, E_{i,i+2}], & i=j-1 \\ -[E_{i,i+1}, E_{i-1,i+1}], & i=j+1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= 0 \quad \forall\, i \neq j$$

$\Rightarrow \forall\, i \neq j$

(1) $(\operatorname{ad}_{e_i})^{-a_{ij}+1} e_j = 0$

Similarly,

2 $\quad (\operatorname{ad} f_i)^{-a_{ij}+1} \cdot f_j = 0 \quad \forall i \neq j$

Relations 1 & 2 are called Serre relations.

~~Thm (Serre)~~

Cartan matrix:

$$A = (a_{ij})_{n-1,n-1} \text{ satisfying}$$

(1) $a_{ii} = 2$

(2) $a_{ij} \leq 0 \quad \forall i \neq j$

(3) $a_{ij} = 0 \iff a_{ji} = 0$

(4) A is positive definite.

Thm (Serre) Let $A = (a_{ij})_{(n-1)\times(n-1)}$ be a Cartan matrix and $\mathfrak{g} = \mathfrak{g}(A)$ be the Lie algebra generated by

$$\{e_i, f_i, h_i \mid 1 \leq i \leq n-1\}$$

satisfying following relations:

(1) $[h_i, h_j] = 0 \quad \forall i, j$

(2) $[h_i, e_j] = a_{ij} e_j, \quad \forall i, j$

(3) $[h_i, f_j] = -a_{ij} f_j, \quad \forall i, j$

(4) $[e_i, f_j] = \delta_{ij} h_i, \quad \forall i, j$

(5) $(ad_{e_i})^{-a_{ij}+1} e_j = 0, \quad \forall i \neq j$

(6) $(ad_{f_i})^{-a_{ij}+1} f_j = 0, \quad \forall i \neq j.$

Then $\mathfrak{g} = \mathfrak{g}(A)$ is a finite dimensional Lie algebra.

In particular, if $A = (a_{ij})$ is indecomposable (i.e. $\nexists$ permutation σ such that $\left(a_{\sigma(i),\sigma(j)}\right) = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \\ & & & A_k \end{pmatrix}, k \geq 2$)

then $\mathfrak{g} = \mathfrak{g}(A)$ is a simple Lie algebra.

In particular, if

$$A = A_{n-1} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & -1 \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

is the Cartan matrix, then

$$\mathfrak{g} = \mathfrak{g}(A) \cong \mathfrak{sl}(n, F).$$

Let $A = (a_{ij})_{n \times n}$ be any Cartan matrix. Define the <u>Dynkin diagram</u> $\Gamma = \Gamma(A)$ as follows:

(1) Γ has n nodes: $\alpha_1, \alpha_2, \cdots, \alpha_n$

(2) ith and jth node are connected by $a_{ij} \cdot a_{ji}$ number of edges.

(3) If $|a_{ij}| > 1$, then there is an arrow pointing to the ith node.

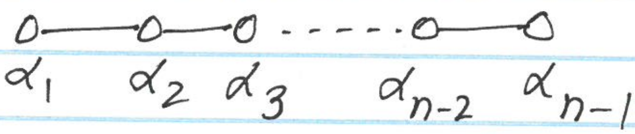
Examples:

(1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = A_2$

Γ :

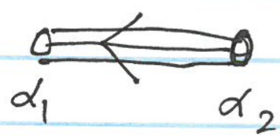
(2) $A = A_{n-1} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$

Γ :



(3) $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Γ :



Geometry of the root system for $\mathfrak{g} = \mathfrak{sl}(n, F)$:

$$\mathfrak{h} = \text{span}_F \{h_1, h_2, \ldots, h_{n-1}\}$$
$$\mathfrak{h}^* = \text{span}_F \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$$

$$\alpha_j = \varepsilon_j - \varepsilon_{j+1}, \quad 1 \le j \le n-1$$

Consider

$$R = \operatorname{span}_{\mathbb{R}}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\} \cong \mathbb{R}^n$$

$$(\varepsilon_i \mid \varepsilon_j) = \delta_{ij}$$

$(\ \mid\ )$ is an inner product on R

$\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ orthonormal basis of R.

$\underbrace{\mathbb{R}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)}_{\shortparallel}$ is a subspace

$$\operatorname{span}_{\mathbb{R}}\{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\}$$

$$\mathfrak{h}^*_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$$

$$\cancel{(\alpha_j \mid \varepsilon_i) = (\varepsilon_j - \varepsilon_{j+1} \mid \varepsilon_i)}$$

$$(\alpha_j \mid \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) = (\varepsilon_j - \varepsilon_{j+1} \mid \varepsilon_1 + \cdots + \varepsilon_n)$$

$$= 0.$$

$\Longrightarrow \mathfrak{h}^*_{\mathbb{R}}$ is the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n\}$

(i.e. $\mathfrak{h}^*_{\mathbb{R}} \cong R/\mathrm{span}_{\mathbb{R}}\{\varepsilon_1 + \cdots + \varepsilon_n\}$)