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$F = \mathbb{C}$ (or $\mathbb{R}$)

$$\mathfrak{g} = sl(n,F) = \{A = (a_{ij})_{n\times n} \in F^{n\times n} \mid tr(A) = 0\}$$

$$[A,B] = AB - BA$$

$sl(n,F)$ simple Lie algebra.

$$\{E_{ii} - E_{i+1,i+1},\ E_{jk} \mid 1 \le i \le n-1,\ 1 \le j \ne k \le n\}$$

basis.

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

$$\mathfrak{h} = \{diag(a_1, a_2, \cdots, a_n) \in sl(n,F) \mid a_1 + \cdots + a_n = 0\}$$

is an abelian subalg., in fact it is maximal.

$\mathfrak{h}$ called the Cartan subalgebra of $\mathfrak{g}$.

$$ad : \mathfrak{h} \longrightarrow gl(sl(n,F))$$

$$x \in \mathfrak{h} \Rightarrow x = \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \ \ x_n \end{pmatrix}$$

$$[x, E_{ij}] = (x_i - x_j) E_{ij}$$

$\varepsilon_i \in \mathfrak{h}^*$ by $\varepsilon_i(x) = x_i$, $1 \le i \le n$

Note $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$, since $x_1 + \cdots + x_n = 0$.

$\alpha_{ij} \in \mathfrak{h}^*$, $\alpha_{ij} = \varepsilon_i - \varepsilon_j$

$\Rightarrow$ $\alpha_{ij}(x) = x_i - x_j$

$$[x, E_{ij}] = \alpha_{ij}(x) E_{ij} \qquad (*)$$

<u>Defn</u>: $0 \neq \alpha \in \mathfrak{h}^*$ is a <u>root</u> if

$$\mathfrak{g}_\alpha = \{A \in \mathfrak{g} \mid [x; A] = \alpha(x) A \ \forall x \in \mathfrak{h}\} \neq 0$$

($\mathfrak{g}_\alpha$ called the α-root space).

$(*) \Rightarrow \Delta = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}$ are the roots

and $\mathfrak{g}_{\alpha_{ij}} = \operatorname{span}\{E_{ij}\}$.

$\Rightarrow$ $\dim \mathfrak{g}_{\alpha_{ij}} = 1$.

Define $\alpha_i = \varepsilon_i - \varepsilon_{i+1} = \alpha_{i,i+1}$, $1 \leq i \leq n-1$

$\Pi = \{\alpha_i \mid 1 \leq i \leq n-1\}$ called the simple roots.

$\{\alpha_i \mid 1 \leq i \leq n-1\}$ basis for $\mathfrak{h}^*$.

For $i < j$,

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \dots + (\varepsilon_{j-1} - \varepsilon_j)$$

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}, \quad 1 \le i < j \le n$$

$\Delta_+ = \{\alpha_{ij} \mid 1 \le i < j \le n\}$ = set of positive roots

for $i > j$, $\alpha_{ij} = \varepsilon_i - \varepsilon_j = -(\varepsilon_j - \varepsilon_i) = -\alpha_{ji}$

$\Delta_- = \{\alpha_{ij} \mid i > j\} = -\Delta_+$ is called the set of negative roots.

$\Rightarrow \quad \Delta = \Delta_+ \cup \Delta_-$ (disjoint union)

$$\mathfrak{g}_0 = \{A \in \mathfrak{g} \mid \underset{\forall x \in \mathfrak{h}}{[x, A] = 0}\} = \mathfrak{h}$$

$\alpha, \beta \in \Delta$

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = ?$

$A \in \mathfrak{g}_\alpha, \quad B \in \mathfrak{g}_\beta$.

$\Rightarrow \forall x \in \mathfrak{h}, \quad [x, A] = \alpha(x) A, \ [x, B] = \beta(x) B$

$$\begin{aligned}
\Rightarrow [x, [A, B]] &= ad_x [A, B] \\
&= [ad_x(A), B] + [A, ad_x(B)] \\
&= \alpha(x)[A, B] + \beta(x)[A, B] \\
&= (\alpha + \beta)(x)[A, B]
\end{aligned}$$

$\Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

So if $\alpha+\beta \notin \Delta \cup \{0\}$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$

Recall $\Delta = \{\pm(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}) \mid 1 \le i < j \le n\}$

If $\alpha \in \Delta_+$, $\cancel{\beta \in \Delta_+}$ ~~and~~ $\cancel{[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq \{0\}}$
~~then~~ $\cancel{\beta = -\alpha}$ $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$.

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\beta$$

are nilpotent subalgebras of $\mathfrak{g}$.

Note $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

called the <u>triangular decomposition of $\mathfrak{g}$</u>.

Consider $\underline{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is a solvable subalgebra called the <u>Borel subalgebra</u> of $\mathfrak{g}$.

<u>Defn</u>: $ht : \mathfrak{h}^* \longrightarrow F$ called <u>height</u> defined by:

$$\alpha \in \mathfrak{h}^* \Rightarrow \alpha = \sum_{i=1}^{n-1} a_i \alpha_i \ , \quad a_i \in F$$

$$ht(\alpha) = \sum_{i=1}^{n-1} a_i \ .$$

In particular,

$$ht(\alpha_{ij}) = j-i \ , \qquad 1 \leq i \neq j \leq n.$$

$$ht(\alpha_{1n}) = n-1$$

$\alpha_{1n} = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ called the highest root of $\mathfrak{g}$.

$$\mathfrak{h} = \text{span}\{E_{ii} - E_{i+1,i+1} = h_i \mid 1 \leq i \leq n-1\}$$

$\check{\alpha}_i = h_i$ are called the coroots and $\check{\Pi} = \{\check{\alpha}_1, \check{\alpha}_2, \cdots, \check{\alpha}_{n-1}\}$ is the set of simple coroots.

$$e_i = E_{i,i+1} \ , \qquad f_i = E_{i+1,i} \ , \quad 1 \leq i \leq n-1$$

For $i<j$, $E_{ij} = [E_{i,i+1}, \cdots [E_{j-2,j-1}, E_{j-1,j}] \cdots]$

$$= [e_i, \cdots [e_{j-2}, e_{j-1}] \cdots]$$

Similarly, for $i > j$, E_{ij} can be obtained by bracketing f_i's.

$\Rightarrow \{h_i, e_i, f_i \mid 1 \le i \le n-1\}$ generate $\mathfrak{g} = sl(n, F)$, called the set of Chevalley generators of $\mathfrak{g}$.

In deed,

$$[h_i, h_j] = 0$$

$$[h_i, e_j] = \alpha_j(h_i) e_j$$

$$[h_i, f_j] = -\alpha_j(h_i) f_j$$

$$[e_i, f_j] = \delta_{ij} h_i$$

For example,

$$[h_i, f_j] = [E_{ii} - E_{i+1,i+1}, E_{j+1,j}]$$

$$= \delta_{i,j+1} E_{ij} - \delta_{i,j} E_{j+1,i} - \delta_{i,j} E_{i+1,j} + \delta_{i,j-1} E_{j+1,i+1}$$

$$= \begin{cases} -2 f_j, & \text{if } i = j \\ f_j, & \text{if } i = j \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

$$-\alpha_j(h_i)\, f_j = -(\varepsilon_j - \varepsilon_{j+1})(E_{ii} - E_{i+1,i+1})\, f_j$$

$$= \begin{cases} -2f_j & , \text{ if } i=j \\ f_j & , \text{ if } i=j\pm 1 \\ 0 & , \text{ otherwise} \end{cases}$$

$$\Rightarrow \quad [h_i, f_j] = -\alpha_j(h_i)\, f_j \, .$$