MA407 TEST 3 Department of Mathematics
Friday, November 17, 2017 NCSU

SHOW YOUR WORK. NO WORK = NO CREDIT.

1. Consider the group $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

   (a) (15%) How many elements of order 3 does G have? List them.

   (b) (10%) How many subgroups of order 3 does G have? Justify your answer.

2. (i) (10%) Consider the cyclic subgroup $H =< (123) >$ of the symmetric group S_3. Is H a normal subgroup of S_3? Justify your answer.

   (ii) (15%) Consider the subgroups $H =< 3 >$ and $K =< 12 >$ of the group $\mathbb{Z}$. List the elements of the factor group H/K and determine their orders.

3. (i)(10%) Show that the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $\varphi(a, b) = a + b$ for all $a, b \in \mathbb{Z}$ is a homomorphism of groups. What is the kernel $ker(\varphi)$?

   (ii)(15%) How many group homomorphisms are there from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{10}$? How many of these are onto? Justify your answer.

4. (i)(15%) Consider abelian group $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$. Is $U(15)$ isomorphic to $\mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$? Justify your answer.

   (ii)(10%) Find all Abelian groups (up to isomorphism) of order 72.

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Answers

#1. (a) $(a,b) \in \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $o(a,b) = \text{lcm}\{o(a), o(b)\} = 3$

$o(a)$	$o(b)$	# of elements
1	3	2
3	1	2
3	3	4

$\therefore \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has 8 elements of order 3.

(b) Each subgroup of order 3 contains 2 elements of order 3.
$\therefore \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has $\frac{8}{2} = 4$ subgroups of of order 3 each of which are cyclic.

#2. (i) $H = \langle (1\,2\,3) \rangle = \{(1), (1\,2\,3), (1\,3\,2)\}$

$\Rightarrow |H| = 3$

$\Rightarrow i_{S_3}(H) = \dfrac{|S_3|}{|H|} = \dfrac{6}{3} = 2$

$\Rightarrow H \trianglelefteq S_3$.

(ii) $H = \langle 3 \rangle = 3\mathbb{Z}$, $K = \langle 12 \rangle = 12\mathbb{Z}$

$\Rightarrow H/K = \{K, 3+K, 6+K, 9+K\}$

$o(K) = 1$, $o(3+K) = 4$, $o(6+K) = 2$
$o(9+K) = 4$

#3. (i) Let $(a,b),(c,d) \in \mathbb{Z} \oplus \mathbb{Z}$

$$\varphi((a,b)+(c,d)) = \varphi(a+c, b+d)$$
$$= (a+c)+(b+d) = (a+b)+(c+d)$$
$$= \varphi(a,b) + \varphi(c,d)$$

$\Rightarrow$ φ is a homomorphism.

$$\ker\varphi = \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid \varphi(a,b) = 0\}$$
$$= \{(a,b) \in \mathbb{Z}\oplus\mathbb{Z} \mid a+b=0\}$$
$$= \{(a,-a) \mid a \in \mathbb{Z}\}$$

(ii) $\varphi : \underset{\langle 1 \rangle}{\underset{\shortparallel}{\mathbb{Z}_{20}}} \longrightarrow \mathbb{Z}_{10}$ is completely determined by $\varphi(1)$ and $o(\varphi(1)) \mid o(1) = 20$.

$$\Rightarrow \varphi(1) = \begin{cases} 0 & \\ 1 & \leftarrow \text{onto} \\ 2 & \\ 3 & \leftarrow \text{onto} \\ 4 & \\ 5 & \\ 6 & \\ 7 & \leftarrow \text{onto} \\ 8 & \\ 9 & \leftarrow \text{onto} \end{cases}$$

$\therefore$ There are 10 homomorphisms, out of which 4 are onto.

#4 (i) $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$o(1) = 1$, $o(2) = 4$, $o(4) = 2$, $o(7) = 4$, $o(8) = o(8^{-1}) = o(2) = 4$
$o(11) = 2$, $o(13) = o(13^{-1}) = o(7) = 4$, $o(14) = 2$

Since $U(15)$ does not have an element of order 8, $U(15) \not\approx \mathbb{Z}_8$

Since $U(15)$ has elements of order 4, $U(15) \not\approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$\Rightarrow U(15) \approx \mathbb{Z}_4 \oplus \mathbb{Z}_2$

(ii) $|G| = 72 = 2^3 3^2$

$\Rightarrow G \approx G_1 \oplus G_2$ where $|G_1| = 2^3$, $|G_2| = 3^2$

Isomorphism classes for G_1:

$3 = 3 \longrightarrow \mathbb{Z}_8$
$\;\; = 2+1 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_2$
$\;\; = 1+1+1 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Isomorphism classes for G_2:

$2 = 2 \longrightarrow \mathbb{Z}_9$
$\;\; = 1+1 \longrightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$\therefore$ G is isomorphic to one of the following:

$\mathbb{Z}_8 \oplus \mathbb{Z}_9$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$