MA407 | TEST 2 | Department of Mathematics
Wednesday, October 18, 2017 | | NCSU

SHOW YOUR WORK. NO WORK = NO CREDIT.

1. (20%) Let $G = GL(2,\mathbb{R})$ and $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid ac \neq 0 \right\}$. Is H a subgroup of G? Justify your answer.

2. (20%) List all cyclic subgroups of the Dihedral group $D_4 = \{I, R, R^2, R^3, H, HR, HR^2, HR^3\}$. Does D_4 have any noncyclic subgroups? If so, list them.

3. (20%) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$.

   (a) Write α^{-1}, $\beta\alpha$, and $\alpha\beta$ as product of disjoint cycles and determine their order.

   (b) Write α^{-1}, $\beta\alpha$, and $\alpha\beta$ as product of transpositions. State whether they are even or odd permutations?

4. (20%) Let G be a group and $a \in G$. Consider the map $f : G \longrightarrow G$ given by $f(x) = a^{-1}xa$ for all $x \in G$. Prove that the map f is an automorphism.

5. (20%) Let G be a group, $H \neq G$ a subgroup of G and $K \neq \{e\}$ a subgroup of H.

   (a) Suppose $|G| = 35$. Show that H is cyclic.

   (b) If $|K| = 10$ and $|G| = 100$, then what are the possible orders of H? Justify your answer.

MA A07
Test 2 – Answers

#1. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in H \Rightarrow ac \neq 0, a'c' \neq 0$

$$AB^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a' & -b'/a'c' \\ 0 & 1/c' \end{pmatrix}$$

$$= \begin{pmatrix} a/a' & -\frac{ab'}{a'c'} + \frac{b}{c'} \\ 0 & c/c' \end{pmatrix} \in H$$

since $\left(a/a'\right)\left(c/c'\right) = (ac)/(a'c') \neq 0$.

$\Rightarrow$ H is a subgroup.

#2. $\langle I \rangle = \{I\}$, $\langle R \rangle = \{I, R, R^2, R^3\} = \langle R^3 \rangle$,

$\langle R^2 \rangle = \{I, R^2\}$, $\langle H \rangle = \{I, H\}$, $\langle HR \rangle = \{I, HR\}$

$\langle HR^2 \rangle = \{I, HR^2\}$ and $\langle HR^3 \rangle = \{I, HR^3\}$ are the cyclic subgroups of D_4.

$\{I, R^2, H, HR^2\}$ and $\{I, R^2, HR, HR^3\}$ are noncyclic subgroups of D_4.

#3. (a) $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 1 & 2 & 3 & 4 & 8 & 6 & 7 \end{pmatrix} = (15432)(687)$

$\Rightarrow o(\alpha^{-1}) = 15$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 7 & 6 & 1 & 2 & 4 & 5 \end{pmatrix} = (13746285)$$

$\Rightarrow o(\beta\alpha) = 8$

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 1 & 3 & 5 \end{pmatrix} = (1\,2\,4\,8\,5\,7\,3\,6)$$

$\Rightarrow o(\alpha\beta) = 8$

(b) $\alpha^{-1} = (12)(13)(14)(15)(67)(68)$, even

$\beta\alpha = (15)(18)(12)(16)(14)(17)(13)$, odd

$\alpha\beta = (16)(13)(17)(15)(18)(14)(12)$, odd.

#4. Let $x, y \in G$

$f(xy) = a^{-1}(xy)a = (a^{-1}xa)(a^{-1}ya) = f(x)f(y)$

$\Rightarrow$ f is a hom.

$f(x) = f(y) \Rightarrow a^{-1}xa = a^{-1}ya \Rightarrow x = y$

$\Rightarrow$ f is 1-1.

For $x \in G$, $axa^{-1} \in G$ and

$f(axa^{-1}) = a^{-1}(axa^{-1})a = x$

$\Rightarrow$ f onto

$\Rightarrow$ f is an automorphism.

#5.(a) $\{e\} \neq H \neq G$ and $|H| \,\big|\, |G| = 35$

$\Rightarrow |H| = 5$ or 7 $\Rightarrow$ H cyclic since 5 & 7 are prime.

(b) $|K| = 10 \,|\, |H|$ & $|H| \,\big|\, 100 = |G|$ and $\{e\} \neq H \neq G$

$\Rightarrow$ $|H| = 20$ or 50 or 10 (if $K = H$).