MA407 TEST 1 Department of Mathematics
Friday, Sept.15, 2017 NCSU

SHOW YOUR WORK. NO WORK = NO CREDIT.

1. (14%) Show that $3n + 2$ and $5n + 3$ are relatively prime for all integers n.

2. (18%) Use the Principle of Mathematical Induction to show that $n^3 - n$ is divisible by 6 for $n \geq 1$.

3. (18%) For $m, n \in \mathbb{Z}$, define $m \sim n$ if $(m - n)$ is even. Show that $\sim$ is an equivalence relation on $\mathbb{Z}$. What are the distinct equivalence classes?

4. (18%) Describe the group G of symmetries of a nonsquare rectangle. Construct the Cayley table for the group G. Is G abelian? Justify your answer.

5. (18%) List the elements in $U(15)$. Find the order of each element in $U(15)$. Is U(15) a cyclic group? Justify your answer.

6. (14%) Consider the group $G = GL(2, \mathbb{Z}_7)$ under matrix multiplication. Find the inverse of $A = \begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$ in G.

MA 407 - Test #1
(Answers)

#1. $(5)(3n+2)+(-3)(5n+3)=1$

$\Rightarrow \gcd(3n+2, 5n+3)=1$

#2. $n=1$: $6 \mid (1)^3-1$ true

Assume $6 \mid (n^3-n)$

$$(n+1)^3-(n+1) = n^3+3n^2+3n+1-n-1$$
$$= (n^3-n)+3n^2+3n$$
$$= (n^3-n)+3n(n+1)$$

Since n or $n+1$ is even, $2 \mid n(n+1)$

$\Rightarrow 6 \mid 3n(n+1)$. Also $6 \mid (n^3-n)$

$\Rightarrow 6 \mid (n^3-n)+3n(n+1)$

$\Rightarrow 6 \mid (n+1)^3-(n+1)$

$\therefore$ By Mathematical Induction we have

$6 \mid (n^3-n)$ for all $n \geq 1$.

#3. $m-m=0$ even $\Rightarrow m \sim m$

$m \sim n \Rightarrow (m-n)$ even $\Rightarrow (n-m)=-(m-n)$ even

$\Rightarrow n \sim m$

$m \sim n$ and $n \sim r \Rightarrow m-n$ & $n-r$ are even

$\Rightarrow m-r=(m-n)+(n-r)$ even

$\Rightarrow m \sim r$.

$\therefore \sim$ is an equivalence relation.

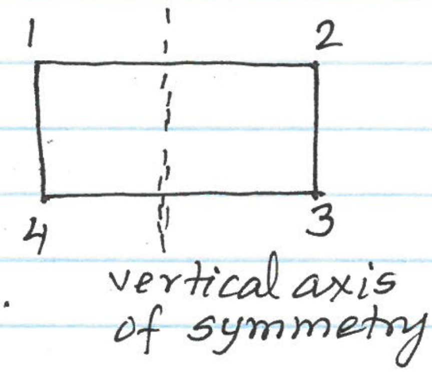
$[0]=\{m \in \mathbb{Z} \mid m \sim 0\}=\{m \in \mathbb{Z} \mid m-0 \text{ even}\}$

$=$ set of even integers.

$[1] = \{m \in \mathbb{Z} \mid m \sim 1\} = \{m \in \mathbb{Z} \mid (m-1) \text{ even}\}$
$= \{m \in \mathbb{Z} \mid m \text{ odd}\} =$ set of odd integers.

$\therefore$ $[0]$ and $[1]$ are the distinct equivalence classes.

#4.



$R =$ rotation about $180°$
$H =$ reflection about vertical axis of symmetry.

Then

$$G = \{I, R, H, HR\} \quad (\text{Note } RH = HR)$$

	I	R	H	HR
I	I	R	H	HR
R	R	I	HR	H
H	H	HR	I	R
HR	HR	H	R	I

$\Rightarrow$ G abelian since the Cayley table is symmetric about the diagonal.

#5. $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$

$o(1) = 1$

$2^2 = 4,\ 2^3 = 8,\ 2^4 = 1 \Rightarrow o(2) = 4 = o(2^{-1}) = o(8)$

$4^2 = 1 \Rightarrow o(4) = 2$

$7^2 = 4,\ 7^4 = 4^2 = 1 \Rightarrow o(7) = 4 = o(7^{-1}) = o(13)$

$11^2 = 1 \Rightarrow o(11) = 2$

$14^2 = 1 \Rightarrow o(14) = 2$

$\Rightarrow U(15)$ is not cyclic since no element has order 8.

#6 $\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix} 3 & -2 \\ -4 & 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix}$

$= 5^{-1}\begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix} = 3\begin{pmatrix} 3 & 5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 6 \end{pmatrix}.$