9/22

Page 86 #13

$\mathbb{Z}_{24} = \langle 1 \rangle$

$\langle 21 \rangle = \{21, 18, 15, 12, 9, 6, 3, 0\} = \langle 3 \rangle$

$\langle 10 \rangle = \{10, 20, 6, 16, 2, 12, 22, 8, 18, 4, 14, 0\}$

$= \langle 2 \rangle$

$\langle 21 \rangle \cap \langle 10 \rangle = \{0, 6, 12, 18\} = \langle 6 \rangle$

$G = \langle a \rangle \quad o(a) = 24 \Rightarrow a^{24} = e$

$\langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^{6} \rangle$

$\overset{\shortparallel}{a^{210}} = a^{\operatorname{lcm}\{21,10\}}$

$\langle a^m \rangle \cap \langle a^n \rangle = \langle a^{\operatorname{lcm}\{m,n\}} \rangle.$

P 70 #18 $\quad a \in G, \quad a^6 = e$

$o(a) = 1 \Leftrightarrow a = e$

$a \neq e, \quad o(a) = 2, 3$ or 6.

Symmetric Group:

$S = \{1, 2, 3, \cdots, n\}$

$P(S) = \{$ set of all 1-1 and onto maps from S to S $\}$

$= S_n$

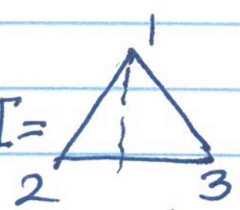
called symmetric group of degree n.

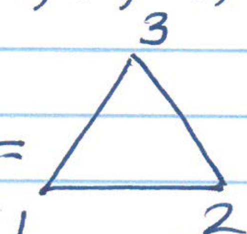
$|S_n| = ?$

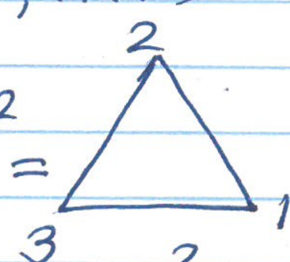
$n(n-1)(n-2) \cdots\cdots 1 = n!$

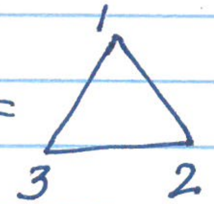
$\Rightarrow |S_n| = n!$

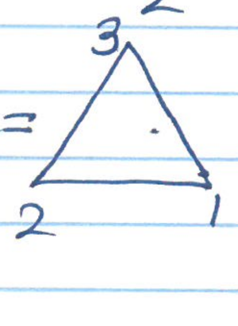
Ex(1) $D_3 = \{I, R, R^2, H, HR, HR^2\}$

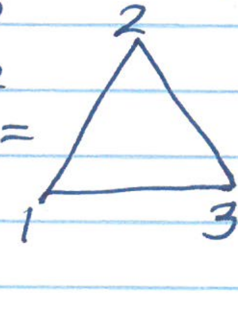
$I =$ 

, $R =$ 

, $R^2 =$ 

$H =$ 

, $HR =$ 

, $HR^2 =$ 

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad HR = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad HR^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$D_3 = S_3 = \{(1), (1\,2\,3), (1\,3\,2), (2\,3), (1\,3), (1\,2)\}$$

In general we view D_n as a subgroup of S_n.

9/25

$$|S_n| = n!$$

$\alpha \in S_n$ called a permutation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

Ex(1) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix} \in S_6$

$$\alpha^{-1} = \begin{pmatrix} 3 & 2 & 5 & 4 & 1 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 4 & 3 & 6 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 4 & 3 & 6 \end{pmatrix}$$

$$\alpha^3 = \alpha^2 \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 1 & 4 & 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = e$$

$\Rightarrow o(\alpha) = 3.$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix}$$

$$= (1\,3\,5)\underbrace{(2)(4)(6)}_{\text{1-cycles}}$$

$$= \underbrace{(1\,3\,5)}_{\text{3-cycle}}$$

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = (1)$$

<u>Ex(2)</u> $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 1 & 7 & 5 & 8 & 3 \end{pmatrix} \in S_8$

$$= \underbrace{(1\,2\,4)}_{\text{3-cycle}}\underbrace{(3\,6\,5\,7\,8)}_{\text{5-cycle}}$$ cycle decomposition

$$= \beta_1 \beta_2$$

$\beta_1 = (1\,2\,4)$, $\beta_2 = (3\,6\,5\,7\,8) \in S_8$

are disjoint cycles $\Rightarrow$ $\beta_1\beta_2 = \beta_2\beta_1$

$\beta_1^2 = (1\,4\,2)$, $\beta_1^3 = (1) = e$

$\Rightarrow$ $o(\beta_1) = 3$.

Similarly, $o(\beta_2) = 5$.

Thm: (1) A k-cycle $(i_1\, i_2 \cdots i_k)$ has order k.

(2) If α, β are disjoint cycles then
$\alpha\beta = \beta\alpha$.

(3) If α, β are disjoint cycles,
then $o(\alpha\beta) = \operatorname{lcm}\{o(\alpha), o(\beta)\}$.

Ex (3) $\beta_1 = (1\,2\,4)$, $\beta_2 = (3\,6\,5\,7\,8)$

$$o(\beta_1) = 3, \quad o(\beta_2) = 5$$

β_1 & β_2 disjoint so we have

$$o(\beta_1\beta_2) = \operatorname{lcm}\{o(\beta_1), o(\beta_2)\} = 15$$

Recall from Ex (2)

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 1 & 7 & 5 & 8 & 3 \end{pmatrix} = \beta_1\beta_2$$

$\Rightarrow o(\alpha) = 15$.

Ex (4) $(1\,2\,4) = (2\,4\,1) = (4\,1\,2)$

$$(i_1\, i_2 \cdots i_k) = (i_1\, i_k)(i_1\, i_{k-1}) \cdots (i_1\, i_2)$$

is a product of 2-cycles (not disjoint)
2-cycles are called transpositions.

Page 88 #38

$m, n \in \mathbb{Z}$

$\langle m \rangle = m\mathbb{Z}$, $\langle n \rangle = n\mathbb{Z}$

$\langle m \rangle \cap \langle n \rangle = \langle k \rangle$ where $k = \text{lcm}\{m, n\}$.

Page 90 #70

$$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} n \in \mathbb{Z} \right\}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in H \quad , \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = ?$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-2} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\right)^2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-k} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k} \middle| k \in \mathbb{Z} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \middle| k \in \mathbb{Z} \right\}$$

$$= H$$

<u>P87 #22</u> $|G| = 3$

$\Rightarrow G = \{e, a, b\}, \quad a \neq b$

$ab \in G \Rightarrow ab = \begin{cases} e \\ a \times \\ b \times \end{cases} \Rightarrow ab = e$

$ab = a \Rightarrow b = e \Rightarrow \Leftarrow$

$ab = b \Rightarrow a = e \Rightarrow \Leftarrow$

$\Rightarrow b = a^{-1}$

$\Rightarrow G = \{e, a, a^{-1}\}$

$$a^2 \in G \Rightarrow a^2 = \begin{cases} e & \times \\ a & \times \\ a^{-1} \end{cases}$$

$$a^2 = e \Rightarrow a = a^{-1} \Rightarrow \Leftarrow$$

$$a^2 = a \Rightarrow a = e \Rightarrow \Leftarrow$$

$\Rightarrow a^2 = a^{-1} \Rightarrow a^3 = e$

$\Rightarrow o(a) = 3$ and $G = \{e, a, a^2\} = \langle a \rangle$.

$\therefore$ <u>G is cyclic.</u>

$(i_1 i_2 \cdots i_k) \in S_n$ $\quad k$-cycle.

$$(i_1 i_2 \cdots i_k) = \underbrace{(i_1 i_k)(i_1 i_{k-1}) \cdots (i_1 i_2)}_{k-1}$$

$\alpha \in S_n$ any permutation

$\alpha = \alpha_1 \alpha_2 \cdots \alpha_m$ disjoint cycles

$=$ product of transpositions (not disjoint).

<u>Defn:</u>

(1) $\alpha \in S_n$ is called "odd" if α is a product of odd number of transpositions.

(2) $\alpha \in S_n$ is called "even" if α is a product of even number of transpositions.

Thm: Any permutation $\alpha \in S_n$ is either odd or even.

Ex(1) $\alpha = (2\,3) \in S_3$ odd

$$\alpha = (1\,3)(1\,2)(1\,3) = (2\,3)$$

Ex(2) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 1 & 7 & 5 & 8 & 3 \end{pmatrix} \in S_8$

Is α even or odd?

$$\alpha = (1\,2\,4)(3\,6\,5\,7\,8)$$
$$= (1\,4)(1\,2)(3\,8)(3\,7)(3\,5)(3\,6)$$

$\Rightarrow \alpha$ is even.

$\alpha, \beta \in S_n$

- α even β even $\Rightarrow \alpha\beta$ even
- α even β odd $\Rightarrow \alpha\beta$ odd
- α odd β even $\Rightarrow \alpha\beta$ odd
- α odd β odd $\Rightarrow \alpha\beta$ even

HW P112 #1-9, 11, 16, 19, 26, 32, 40, 42.