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P220 #7

$$|G| = 108 = 2^2 3^3$$

$G \approx G_1 \oplus G_2 \qquad |G_1| = 2^2, \; |G_2| = 3^3$

Possibilities for G_1:

$2 = 2 \;:\; \mathbb{Z}_4$

$\;\; = 1+1 : \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Possibilities for G_2:

$3 = 3 \;:\; \mathbb{Z}_{27}$

$\;\; = 2+1 : \mathbb{Z}_9 \oplus \mathbb{Z}_3$

$\;\; = 1+1+1 : \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$G \approx \mathbb{Z}_4 \oplus \mathbb{Z}_{27}$ (2 elts of order 3)

or $\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$ (8 elts of order 3)

or $\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ (26 elts of order 3)

or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{27}$ (2 elts of order 3)

or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$ (8 elts of order 3)

or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. (26 elts of order 3)

The ones with exactly 4 subgroups of order 3 (i.e. exactly 8 elts of order 3) are:

- $\mathbb{Z}_4 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$.

Ring: $(R, +, \cdot)$

Properties: R: ring, $a, b, c \in R$.

(1) $a \cdot 0 = 0 = 0 \cdot a$

(2) $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$

(3) $(-a) \cdot (-b) = a \cdot b$

(4) $a \cdot \underbrace{(b + (-c))}_{= b-c} = a \cdot b - a \cdot c$

(5) If R has a unity $1 \in R$, then

$(-1) \cdot a = -a$ & $(-1) \cdot (-1) = 1$.

Subring: A nonempty subset S of a ring R is a subring if

(1) $\forall\ a, b \in S$, $a - b \in S$

(2) $\forall\ a, b \in S$, $a \cdot b \in S$.

Ex(1) m be any positive integer. Then $m\mathbb{Z}$ is a subring of $\mathbb{Z}$.

Let $mk, ml \in m\mathbb{Z}$

$$mk - ml = m(k-l) \in m\mathbb{Z}$$
$$(mk)\cdot(ml) = m(mkl) \in m\mathbb{Z}.$$

Ex (2) $\{0, 2, 4\} \subset \mathbb{Z}_6$

$-$	0	2	4
0	0	2	4
2	4	0	2
4	2	4	0

$\cdot$	0	2	4
0	0	0	0
2	0	4	2
4	0	2	4

$\Rightarrow$ $\{0, 2, 4\}$ is a subring of $\mathbb{Z}_6$.

HW P232 #2, 6, 9, 13, 18, 19, 20, 21, 23, 26, 28, 36, 40, 41, 42, 43, 46.

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Test 3 covers:

(25pt) Chap 8 : Direct sum

(25pt) Chap 9 : Normal subgroup & Factor group

(25pt) Chap 10 : Homomorphism

(25pt) Chap 11 : Finite abelian group

P168 #22

# of elts of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$

$= 48$

# of elts of order 15 in a cyclic group of order 15

$= |U(15)| = 8$

# of cyclic subgroups of order 15

$$\frac{48}{8} = 6.$$

$(a, b) \in \mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$

$o(a,b) = \operatorname{lcm}\{o(a), o(b)\} = 15$

(15,	1)	8
(3,	5)	8
(15,	5)	32
		48

P187 #1

$H = \{(1), (12)\}$ subgroup of S_3

$(123)H = \{(123)(1), (123)(12)\} = \{(123), (13)\}$

$H(123) = \{(1)(123), (12)(123)\} = \{(123), (23)\}$

$(123)H \neq H(123) \Rightarrow H \ntriangleleft S_3$.

$K = \langle(123)\rangle = \{(1), (123), (132)\}$

$i_{S_3}(K) = \dfrac{|S_3|}{|K|} = \dfrac{6}{3} = 2$

$\Rightarrow K \triangleleft S_3$.

P189 #27

$G = U(16) = \{1, 3, 5, 7, 9, 11, 13, 15\}$

$H = \{1, 15\} = \langle 15 \rangle \approx \mathbb{Z}_2$

$K = \{1, 9\} = \langle 9 \rangle \approx \mathbb{Z}_2$

$\Rightarrow H \approx K$

$|G/H| = 4$

$G/H = \{H, 3H, 5H, 7H\}$

$o(H) = 1,\ o(3H) = 4$

$\Rightarrow\ G/H \approx \mathbb{Z}_4$

$|G/K| = 4$

$G/K = \{K, 3K, 5K, 7K\}$

$o(K) = 1,\ o(3K) = 2,\ o(5K) = 2,\ o(7K) = 2$

$\Rightarrow\ G/K \not\approx \mathbb{Z}_4$

$\Rightarrow\ G/H \not\approx G/K.$

P207 #35

$\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$

$\varphi(a, b) = a - b$

$$\begin{array}{ccc} \varphi((a,b)+(c,d)) & \overset{?}{\underset{\checkmark}{=}} & \varphi(a,b) + \varphi(c,d) \\ \| & & \| \\ \varphi(a+c, b+d) & & a-b+c-d \\ \| & & \\ (a+c)-(b+d) & & \end{array}$$

$\Rightarrow\ \varphi$ is a hom.

$\ker\varphi = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid \underbrace{\varphi(a,b) = 0}_{a = b}\}$

$= \{(a,a) \mid a \in \mathbb{Z}\}$

$$\varphi^{-1}(3) = \{(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \mid \underset{a-b}{\underset{\shortparallel}{\varphi(a,b)}} = 3\}$$

$$= \{(b+3, b) \mid b \in \mathbb{Z}\}$$

$$= (3,0) + \ker\varphi$$

P220 #10

G abelian, $|G| = 360$
$= 2^3 \cdot 3^2 \cdot 5$

$G \approx G_1 \oplus G_2 \oplus G_3$

$|G_1| = 2^3$: $3 = 3 \longrightarrow \mathbb{Z}_8$
$= 2+1 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_2$
$= 1+1+1 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$|G_2| = 3^2$: $2 = 2 \longrightarrow \mathbb{Z}_9$
$= 1+1 \longrightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_3$

$|G_3| = 5$: $G_3 \approx \mathbb{Z}_5$

$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$	$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$

P 233 #23

$\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$

Gaussian integers.

$U(\mathbb{Z}[i]) = ?$

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} + \frac{(-b)i}{a^2+b^2} \in \mathbb{Z}[i]$$

if (1) $a = \pm 1$, $b = 0 \Rightarrow 1, -1$
(2) $a = 0$, $b = \pm 1 \Rightarrow i, -i$

$\therefore U(\mathbb{Z}[i]) = \{1, -1, i, -i\}$.

$(R, +, \cdot)$ ring

$0 \neq a \in R$ is a <u>zero divisor</u> if
$a \cdot b = 0$ for some $0 \neq b \in R$.

<u>Ex(1)</u> What are the zero divisors in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$?

$2 \cdot 3 = 0$, 2 and 3 are zero divisors
$3 \cdot 4 = 0 \Rightarrow 4$ is a zero divisor.

Ex(2) $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$

$0 \neq a \in \mathbb{Z}_n$ zero divisor

$\Rightarrow \exists\ 0 \neq b \in \mathbb{Z}_n$ such that $a \cdot b = 0$

If 'a' is a unit then $a^{-1} \in \mathbb{Z}_n$ and

$$a \cdot a^{-1} = 1 = a^{-1} \cdot a$$

Then $a \cdot b = 0$

$$(a^{-1} \cdot a) \cdot b = a^{-1} \cdot 0$$

$$\underbrace{1 \cdot b}_{=b} = \underbrace{a^{-1} \cdot 0}_{=0}$$

$b = 0$ contradiction

So a zero divisor can not be a unit.

$U(n) =$ set of units in $\mathbb{Z}_n$

$\Rightarrow 0 \neq a \in \mathbb{Z}_n \setminus U(n)$ is a zero divisor.

Defn: A commutative ring with unity is an <u>integral domain</u> if R does not have any zero divisors.

Ex(1) $\mathbb{Z}_6$ is not an integral domain, but $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is an integral domain.

Thm: $\mathbb{Z}_n$ is an integral domain if and only if n is prime.