G group, N be a subgroup.

N is a <u>normal</u> subgroup of G if

(*). $Na = aN$ for all $a \in G$

equivalently (**) $a^{-1}Na = N$ for all $a \in G$

equivalently (***) $a^{-1}Na \subseteq N$ for all $a \in G$.

Notation: $N \triangleleft G$.

<u>Prop</u>: $\underset{\substack{\| \\ \#\text{ of (left) cosets of } N \text{ in } G.}}{i_G(N)} = 2 \Rightarrow N \triangleleft G$.

<u>Pf</u>: $i_G(N) = 2 \Rightarrow G = N \sqcup Na$, $a \in G$

$= N \sqcup bN$, $b \in G$

$\Rightarrow Na = G \setminus N$, $\underset{\substack{\| \\ aN}}{bN} = G \setminus N = Na$

$\Rightarrow N \triangleleft G$.

<u>Ex (1)</u> $G = S_n$, $i_{S_n}(A_n) = \dfrac{|S_n|}{|A_n|} = 2$

$\Rightarrow A_n \triangleleft S_n$.

<u>Ex.(2)</u> $G = D_4$, $\langle R \rangle = \{I, R, R^2, R^3\}$

$i_{D_4}(\langle R \rangle) = 2 \Rightarrow \langle R \rangle \triangleleft D_4$.

Remark: G abelian group & N subgrp
$\Longrightarrow N \trianglelefteq G$.

~~Ex(3)~~ $\cancel{G = \mathbb{Z}_n, \ 2 \leq m < n}$

Ex(3) $\quad G = \mathbb{Z}, \quad m \in \mathbb{Z}_{>0}$

$$\langle m \rangle = m\mathbb{Z} \trianglelefteq \mathbb{Z}$$

since $\mathbb{Z}$ abelian.

Ex(4) . $S_3 = \{(1), (12), (23), (13), (123), (132)\}$

$$\langle (12) \rangle = \{(1), (12)\} = H$$

$$(13)H = \{(13)(1), (13)(12)\} = \{(13), (123)\}$$

$$H(13) = \{(1)(13), (12)(13)\} = \{(13), (132)\}$$

$$(13)H \neq H(13)$$

$\Longrightarrow H \ntrianglelefteq S_3$.

$$\langle (123) \rangle = \{(1), (123), (132)\} = K$$

$$(12)K = \{(12)(1), (12)(123), (12)(132)\}$$

$$= \{(12), (23), (13)\}$$

$$= S_3 \setminus K = K(12)$$

$K \trianglelefteq S_3$.

<u>Ex(5)</u> $G = GL(2,\mathbb{R}) = \left\{ \overset{A}{\overset{\shortparallel}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \,\middle|\, \overset{\det(A)}{\overset{\shortparallel}{ad - bc}} \neq 0 \right\}$

$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad-bc = 1 \right\}$

Recall $\quad A, B \in GL(2,\mathbb{R})$

$$\det(AB) = (\det A)(\det B).$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}), \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2,\mathbb{R})$$

$A^{-1}BA \in SL(2,\mathbb{R})$?

$$\det(A^{-1}BA) = (\det A^{-1})\underset{\underset{1}{\shortparallel}}{(\det B)}(\det A)$$

$$= (\det A^{-1})(\det A) = \det(A^{-1}A) = 1$$

$\Rightarrow \quad A^{-1}BA \in SL(2,\mathbb{R}) \quad \forall\, A \in GL(2,\mathbb{R})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad B \in SL(2,\mathbb{R})$

$\Rightarrow \quad SL(2,\mathbb{R}) \lhd GL(2,\mathbb{R}).$

G group, N normal subgroup of G.

$\Rightarrow \quad Na = aN \quad \forall\, a \in G$

Set $\quad G/N = \{ Na \mid a \in G \}$

Define ⊛ $(Na)(Nb) = N(ab) \quad \forall\, a,b \in G$

Suppose $\quad Na = Na', \; Nb = Nb'$

Need to show: $\quad N(ab) \overset{?}{=} N(a'b')$

$Na = Na' \iff a'a^{-1} \in N$

$Nb = Nb' \iff b'b^{-1} \in N$

$(a'b')(ab)^{-1} = a'(b'b^{-1})a^{-1}$

$= \underbrace{(a'a^{-1})}_{\in N} \overbrace{a\underbrace{(b'b^{-1})}_{\in N}a^{-1}}^{\in N \text{ since } N \trianglelefteq G} \in N$

$\Rightarrow N(ab) = N(a'b')$

$\Rightarrow$ ⊛ is well defined.

$(Na)(N) = (Na)(Ne) = N(ae) = Na$

$\Rightarrow \quad \bar{e} = N = Ne$ is the identity in G/N

$$(Na)(Na^{-1}) = N(aa^{-1}) = Ne = N = \bar{e}$$

$$\Rightarrow (Na)^{-1} = Na^{-1}$$

$$((Na)(Nb))(Nc) \overset{?}{=} (Na)((Nb)(Nc))$$

$$\begin{array}{ccc} \| & & \| \\ N(ab)\,Nc & & (Na)(N(bc)) \\ \| & & \| \\ N(ab)c & = & Na(bc) \end{array}$$

$\therefore$ G/N is a group called the "factor group" of G over N.

<u>Ex(1)</u> $G = \mathbb{Z}$, $N = \langle m \rangle = m\mathbb{Z} \trianglelefteq G$

$$\mathbb{Z}/m\mathbb{Z} = \{\underset{\substack{\| \\ N \\ \| \\ \bar{e}}}{0+N}, 1+N, \cdots, (m-1)+N\}$$

Consider $\varphi : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}_m = \{0, 1, \cdots, m-1\}$

$$\varphi(k + m\mathbb{Z}) = k \in \mathbb{Z}_m$$

φ is a 1-1 and onto map.

$$\varphi((k+m\mathbb{Z}) + (l+m\mathbb{Z})) \overset{?}{=} \varphi(k+m\mathbb{Z}) +_m \varphi(l+m\mathbb{Z})$$

$$\varphi((k+m\mathbb{Z})+(l+m\mathbb{Z}))$$
$$= \varphi((k+l)+m\mathbb{Z}) = (k+l) \bmod m$$
$$= k(\bmod m) +_m l (\bmod m)$$
$$= k +_m l = \varphi(k+m\mathbb{Z}) +_m \varphi(l+m\mathbb{Z})$$

$\Rightarrow$ φ is an isomorphism.

<u>Ex(2)</u> $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $N = \langle (1,1) \rangle$

$N = \{(1,1), (0,2), (1,3), (0,0)\}$

$|G| = 8$, $|N| = 4$

$G/N = \{N, N+(0,1)\}$

G group. $N = Z(G) = \{x \in G \mid xy = yx \ \forall y \in G\}$

<u>Fact:</u> $Z(G) \triangleleft G$ since

$x \in Z(G)$, $y \in G$

$(yx)y^{-1} = (xy)y^{-1} = x(yy^{-1}) = x \in Z(G)$.

$\Rightarrow y\, Z(G)\, y^{-1} \subseteq Z(G)$.

Thm: $G/Z(G)$ cyclic $\Rightarrow$ G abelian

HW: P187 #1, 2, 3, 6, 11, 14, 16, 17,
24, 25, 27, 28

P188 #6

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a,b,d \in \mathbb{R}, ad \neq 0 \right\} \subset GL(2,\mathbb{R})$$

- $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in H$

$$AB^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1/a' & -b'/a'd' \\ 0 & 1/d' \end{pmatrix}$$

$$= \begin{pmatrix} a/a' & -\frac{ab'}{a'd'} + b/d' \\ 0 & d/d' \end{pmatrix} \in H$$

since $(a/a')(d/d') = \frac{ad}{a'd'} \neq 0$

$\Rightarrow H$ subgroup.

- $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H$, $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in GL(2,\mathbb{R})$

$\underset{\det A}{\overset{\shortparallel}{ad}} \neq 0$ $\qquad k = \underbrace{x_1x_4 - x_2x_3}_{= \det X} \neq 0$

$$XAX^{-1} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \frac{1}{k} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$$

$$= \frac{1}{k} \begin{pmatrix} ax_1 & bx_1 + dx_2 \\ ax_3 & bx_3 + dx_4 \end{pmatrix} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}$$

$$= \frac{1}{k}\begin{pmatrix} ax_1x_4 - x_3(bx_1+dx_2) & -ax_1x_2+x_1(bx_1+dx_2) \\ \underbrace{ax_3x_4 - x_3(bx_3+dx_4)}_{\neq 0} & * \end{pmatrix}$$

$\Rightarrow H$ not a normal subgroup.

For example, take $X = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

$$XAX^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b-d \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} b-d & a+b-d \\ \underset{\neq 0}{-b} & a+b \end{pmatrix}$$

$$\notin H$$

G, G' groups.

Recall that a map $\varphi : G \to G'$ is a homomorphism if

$$\forall\, a, b \in G,\ \varphi(ab) = \varphi(a)\varphi(b)$$

<u>Defn:</u> $\{a \in G \mid \varphi(a) = e' \in G'\} = \ker(\varphi)$

called the <u>kernel</u> of φ.

$e \in \ker \varphi$

φ is 1-1 $\iff$ $\ker \varphi = \{e\}$

<u>Claim:</u> $\underbrace{\ker(\varphi)}_{= N} \triangleleft G$.

• N subgroup of G.

Let $x, y \in \ker(\varphi) = N$

$\Rightarrow \varphi(x) = e', \varphi(y) = e'$.

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)(\varphi(y))^{-1}$$
$$= e'(e')^{-1} = e'$$

$\Rightarrow xy^{-1} \in N \Rightarrow N$ subgroup.

• N is a normal subgroup.

$x \in N \Rightarrow \varphi(x) = e'$

$a \in G$

$$\varphi(axa^{-1}) = \varphi(a)\underbrace{\varphi(x)}_{= e'}\varphi(a^{-1}) = \varphi(a)\varphi(a^{-1})$$
$$= \varphi(a)(\varphi(a))^{-1} = e'$$

$\Rightarrow a x a^{-1} \in N \Rightarrow N \trianglelefteq G$.

$G/_{\ker(\varphi)}$ is the factor group.

Fundamental Theorem of Homomorphism:

Let $\varphi : G \longrightarrow G'$ be a homomorphism and it is onto. Then

$$G/_{\ker\varphi} \approx G'.$$

<u>Ex (1)</u> $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_m$ given by

$$\varphi(n) = n \pmod{m}$$

Then φ is a homomorphism. Clearly φ is onto.

$$\ker\varphi = \{ n \in \mathbb{Z} \mid \varphi(n) = 0 \} = m\mathbb{Z}$$

By Hom. Thm.

$$\mathbb{Z}/_{m\mathbb{Z}} \approx \mathbb{Z}_m.$$

11/3

P189 #25

$G = U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$

$|U(32)| = 16$, $o(15) = 2$

$H = \{1, 15\} = \langle 15 \rangle$, $|H| = 2$

G abelian, $H \triangleleft G$

$$\left|G/H\right| = |G|/|H| = 16/2 = 8$$

$$G/H = \{\underset{\overset{\shortparallel}{\bar{e}}}{H}, 3H, 5H, 7H, 17H, 19H, 21H, 23H\}$$

$o(H) = 1$, $o(3H)$

$(3H)(3H) = 9H = \{9, 7\} = 7H$

$(7H)(7H) = 17H$

$(17H)(17H) = 1H = \bar{e}$

$\Rightarrow o(3H) = 8 \Rightarrow G/H \approx \mathbb{Z}_8$

G, G' groups

$\varphi : G \longrightarrow G'$ is a hom. if

$\varphi(ab) = \varphi(a)\varphi(b) \quad \forall\, a, b \in G.$

$$\ker\varphi = \{ x \in G \mid \varphi(x) = e' \}$$

- φ is 1-1 $\iff$ $\ker\varphi = \{e\}$
- $\ker\varphi \triangleleft G$

Hom. Thm: $\varphi : G \to G'$ hom. and onto.

Then $\quad G/_{\ker\varphi} \approx G'$

Example: $\quad G$ group. $N \triangleleft G$

$$\pi : G \longrightarrow G/_N$$

defined by $\quad \pi(x) = xN \quad \forall x \in G.$

$$\pi(xy) = (xy)N = (xN)(yN)$$
$$= \pi(x)\pi(y)$$

$\Rightarrow \pi$ is a hom., called "canonical hom".

Let $aN \in G/_N$. Then $\pi(a) = aN$

$\Longrightarrow \pi$ is onto.

$$\ker\varphi = \{ x \in G \mid \underset{xN}{\overset{\shortparallel}{\pi(x)}} = \bar{e} = N \}$$

$$= N$$

Homomorphism properties:

$$\varphi : G \longrightarrow G' \text{ hom.}$$

- $\varphi(e) = e'$
- $\varphi(a^{-1}) = (\varphi(a))^{-1}$, $a \in G$
- $\varphi(a^n) = (\varphi(a))^n$, $n \in \mathbb{Z}$, $a \in G$
- $\forall a \in G$, $o(\varphi(a)) \mid o(a)$.
- $\varphi(G) = \{\varphi(a) \mid a \in G\}$ subgroup of G'
- G cyclic $\Rightarrow \varphi(G)$ cyclic
- G abelian $\Rightarrow \varphi(G)$ abelian
- $H \trianglelefteq G \Longrightarrow \varphi(H) \trianglelefteq \varphi(G)$.

Ex(2) $\quad \varphi : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ hom

$$1 \longmapsto 2$$

(i.e. $\varphi(1) = 2$)

$$\varphi(\mathbb{Z}_{10}) = \underbrace{\{0, 2, 4, 6, 8\}}_{= \langle 2 \rangle} \neq \mathbb{Z}_{10}$$

$\varphi(6) = 2 = \varphi(1) \Rightarrow \varphi$ not 1-1

$o(1) = 10$, $\quad o(\varphi(1)) = o(2) = 5 \mid 10 = o(1)$.

$\ker \varphi = \{0, 5\}$

$\varphi^{-1}(2) = \{k \in \mathbb{Z}_{10} \mid \varphi(k) = 2\}$

$= \{1, 6\} = 1 + \ker \varphi$

In general,

If $\varphi : G \longrightarrow G'$ hom.,

$N = \ker \varphi$ and $\varphi(a) = x \in G'$

then $\varphi^{-1}(x) = \{y \in G \mid \varphi(y) = x\}$

$= aN.$

HW P205: 3, 8, 9, 11, 14, 15, 20, 24, 25, 26, 33, 35.